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ISOMORPHISMS OF NONCOMMUTATIVE LOG-ALGEBRAS

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ИЗОМОРФИЗМЫ НЕКОММУТАТИВНЫХ ЛОГ-АЛГЕБР

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Abstract. The article establishes a necessary and sufficient condition for the isomorphism of log-algebras constructed on different von Neumann algebras by a faithful normal finite trace.

Аннотация. В статье устанавливается необходимое и достаточное условие изоморфизма лог-алгебр, построенных на различных алгебрах фон Неймана по точному нормальному конечному следу.

Keywords: von Neumann algebra, faithful normal finite trace, log-algebra, isomorphisms.

Ключевые слова: алгебра фон Неймана, точный нормальный конечный след, лог-алгебра, изоморфизмы.

Introduction

Let M be the von Neumann algebra, μ the faithful normal finite trace on M, S(M, μ) — *-algebra of measurable operators associated with M.

Consider the set $L_{log}(M, \mu)=\{T \in S(M, \mu): \mu(\log(1+|T|))<\infty\}$ and the function $||T||_{log}= \mu(\log(1+|T|))$ on $L_{log}(M, \mu)$. In the work ([1] Lemma 4.1 and 4.3.) the following properties of the function $||\cdot||_{log}$ have been proved.

Lemma 1. Let $S, T\in S(M, \mu)$. Then

a) $||T||_{log}>0$, provided $T\neq 0$;

b) $||\alpha T||_{log}\leq ||T||_{log}$ for all scalars α with $|\alpha|\leq 1$;

c) If $T\in L_{log}(M, \mu)$, then

$$\lim_{\alpha\to 0} \| \alpha T \|_{log} = 0$$

d) $||S+T||_{log}\leq ||S||_{log}+||T||_{log}$;

e) $||S\cdot T||_{log}\leq ||S||_{log}+||T||_{log}$.

It follows from properties a), b), c), d) that the function $||\cdot||_{log}$ is an F-norm on the space $L_{log}(M, \mu)$, and property e) imply that the space $L_{log}(M, \mu)$ is a topological algebra with respect to topology generated by the metric $\rho(S,T)=||S-T||_{log}$ ([1], corollary 4.6). Let's call algebra $L_{log}(M, \mu)$ log - algebra.

In the present paper, we determine the necessary and sufficient condition the isomorphism of the log-algebras constructed by various the faithful normal finite trace on various von Neumann algebras.

Ease of Use

Let M be a von Neumann algebra with faithful normal finite traces μ and ν. It follows from the inequality $\log|f(z)|\leq \frac{1}{p}|f(z)|^p$ that $L_p(\Omega,\nu)\subset L_{\log}(\Omega,\nu)$ for $p\in(0,\infty)$. And it follows from the inequalities $k_1\log_a c\leq \log_b c \leq k_2\log_a c$ that the finiteness of the value $\int_\Omega log(1+|f(z)|)d\nu$ does not depend on the choice of the base of the logarithm. Here k_1 is a sufficiently small number and k_2 is a sufficiently large number.

Let μ and ν be faithful normal finite traces on the von Neumann algebra M, denote by $\square = \frac{d\nu}{d\mu}$ the Radon Nikodim derivative of trace ν with respect to μ, such a central positive operator from $L_1(M,\mu)$ for which the equality $\nu(x)=\mu(hx)$ holds for all $x\in M$ [2]. From here we get $\mu(h)=\nu(1)$, i. e. $h\in L_1(M,\mu)$. Moreover, there exists a measurable operator $\square^{-1} = \frac{d\nu}{d\mu}$ [3].

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Proposition 2. $L_{\log}(M,\mu)\subset L_{\log}(M,\nu)$ if and only if $h\in M$.

Proof. Let $h\in M$ and $f\in L_{\log}(M,\mu)$, i.e. $\int_\Omega log(1+|f(z)|)d\mu<\infty$. Since h is central, the algebra of measurable operators generated by the operators h and f will be commutative. Therefore, in this case $S(M,\mu)$ can be identified with the function space on Ω. Then

$$\int_\Omega log(1+|f(z)|)d\nu = \int_\Omega (\square\, log(1+|f(z)|))d\mu \leq$$

$$\leq \|\square\|_\infty \int_\Omega log(1+|f(z)|)d\mu < \infty$$

Hence $f\in L_{\log}(M,\mu)$, i.e. $L_{\log}(M,\mu)\subset L_{\log}(M,\nu)$.

Conversely, let $0<h\in L_1(M,\mu)\backslash M$. Then it is possible to construct an infinite sequence of sets $M_n=\{z\in\Omega:\ n\leq h(z)\leq n+1\}$. Now consider the subset of natural numbers $N_0=\{n\in N:\ \mu(M_n)\}$. Let us redesignate the elements of the set N_0 as follows $N_0=\{n_1,n_2,\ldots\}$, $n_k<n_{k+1}$.

Consider the function

$$g(z) = \frac{1}{k^2\mu(M_{n_k})};\, z\in M_{n_k}$$
$$g(z) = o,\, z\in \Omega\backslash\cup_k M_{n_k}.$$

Let's put $f(z)=e^g-1$, then

$$\int_\Omega log(1+|f(z)|)d\mu = \sum_{k=1}^{\infty}\frac{\mu(M_{n_k})}{k^2\mu(M_{n_k})} = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty \tag{1}$$

However

$$\int_\Omega log(1+|f(z)|)d\mu = \nu(log(1+|f(z)|)) = \mu(\square(z))\, log(1+f(z)) \tag{2}$$

$$= \mu(\square g) \geq \sum_{k=1}^{\infty}\frac{n_k\mu(M_{n_k})}{k^2\mu(M_{n_k})} = \sum_{k=1}^{\infty}\frac{n_k}{k^2} \geq \sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

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From (1) and (2) it follows that $f \in L_{log}(M, \mu)$, and $f \notin L_{log}(M, \mu)$, i.e. $L_{log}(M, \mu)$, is not a subset of $L_{log}(M, \nu)$, for $h \in L_1(M,\mu)\backslash M$. So from $L_{log}(M, \mu) \subset L_{log}(M,\nu)$ if and only if, when $h \in M$.

Let h be the Radon-Nikodim derivative of the faithful normal finite trace of ν with respect to the faithful normal finite trace of μ The von Neumann algebra M is hence finite. Therefore, by virtue of Theorem 1 [4], h and h^{-1} are elements of the algebra of measurable elements. Now from the equality $\nu(x) = \mu(hx)$ we get $\nu(h^{-1}x) = \mu(h^{-1}hx) = \mu(x)$, i.e. h^{-1} is the derivative of Radon-Nikodim of trace μ with respect to ν. Therefore, from Proposition 2 we obtain.

Corollary 3. $L_{log}(M, \mu) = L_{log}(M, \nu)$ if and only if $h, h^{-1} \in M$.

Let M and N be noncommutative von Neumann algebras with faithful normal finite traces μ and ν, respectively. Let $\alpha: M \to N$ be an isomorphism from M to N. Then the functional $\mu \circ \alpha^{-1}$ will be an faithful normal finite trace on N.

Definition 4. Traces μ and ν are said to be equivalent if there exists a ∗-isomorphism $\alpha: M \to N$ such that one of the following equivalent conditions is satisfied:

$$(i)\ L_{log}(N,\nu) = L_{log}(N, \mu \circ \alpha^{-1});$$

$$(ii)\ \frac{d\nu}{d\mu \circ \alpha^{-1}}, \frac{d\mu \circ \alpha^{-1}}{d\nu} \in N.$$

The equivalence of conditions (*i*) and (*ii*) follows from Corollary 3.

Theorem 5. The algebras $L_{log}(M, \mu)$ and $L_{log}(N, \nu)$ are ∗-isomorphic if and only if μ and ν are equivalent.

Proof. Let μ and ν be equivalent, i. e., there exists a ∗-isomorphism $\alpha: M \to N$, for which condition (*i*) is satisfied. The ∗-isomorphism $\alpha: M \to N$, extends to the ∗-isomorphism α′ onto the algebra of measurable functions $L_0(\Omega)$. In this case, using the continuity of α′ with respect to the topology of convergence in measure, we obtain that

$$\alpha'(L_{log}(M,\mu)) = L_{log}(N,\mu \circ \alpha^{-1}). \tag{3}$$

By virtue of condition (*i*), we have

$$L_{log}(N,\mu \circ \alpha^{-1}) = L_{log}(N,\nu). \tag{4}$$

From (3) and (4) we obtain that $L_{log}(M, \mu)$ and $L_{log}(N,\nu)$ are ∗-isomorphic.

Conversely, let α′ be a ∗-isomorphism from $L_{log}(M, \mu)$ to $L_{log}(N,\nu)$. Then α′ translates bounded elements from $L_{log}(M, \mu)$ into bounded elements from $L_{log}(N,\nu)$, i. e. the restriction of α′ to M is a ∗-isomorphism from M to N. Moreover, the ∗-isomorphism from M to N satisfies the condition that traces μ and ν are equivalent.

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